Random matrices with correlated elements: A model for disorder with interactions

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(Received 19 February 2004; revised manuscript received 14 September 2004; published 10 February 2005; publisher error corrected 25 April 2005)

The complicated interactions in the presence of disorder lead to a correlated randomization of states. The Hamiltonian as a result behaves like a multiparametric random matrix with correlated elements. We show that the eigenvalue correlations of these matrices can be described by the single parametric Brownian ensembles. The analogy helps us to reveal many important features of the level statistics in interacting systems, e.g., a critical point behavior different from that of noninteracting systems, the possibility of extended states even in one dimension, and a universal formulation of level correlations.

DOI: 10.1103/PhysRevE.71.026226 PACS number(s): 05.45.−a, 03.65.Sq, 05.40.−a

I. INTRODUCTION

The theoretical formulation of physical properties of electronic systems is usually based on independent electron approximation. However, experiments on quantum dots as well as extended semiconducting electron heterostructures show that Coulomb interaction between electrons is by no means small [1,2]. The influence of Coulomb interaction is particularly strong in the presence of a disordered environment, with bulk and mesoscopic systems both revealing new features [4,7]. Examples include Coulomb blockade phenomena, low energy anomalies in transport and thermodynamics coefficients, non-Fermi liquid effects in effectively one-dimensional structures, various manifestations of the Kondo effects, the fractional quantum Hall effects, etc. A theory of physical properties including electron electron (*e-e*) interaction is therefore very much required.

The presence of interactions along with disorder makes a physical system so complex that its properties can be analyzed only by a statistical tool. In the past, the statistics of single particle states of noninteracting systems with external disorder potential has been well-modeled by the Wigner-Dyson ensembles, also known as standard random matrix ensembles (RMT) [5,3,7]. The ensembles have also been used successfully to describe the statistical properties of the high energy states of many body systems with no disorder, e.g., nuclei, atoms, etc. [5,3,7]. The intuition suggests therefore a possibility of random matrix modeling of systems when disorder and interactions are both present. Recent studies in this direction have led to two types of random matrix models, however both impose specific conditions on nature and degree of *e-e* interactions as well as disorder in the system [8,9]. In this paper, we consider a random matrix model suitable for generic conditions for both disorder and interactions and show that the spectral statistics can be described by a mathematical formulation analogous to that of noninteracting disordered systems. The parameter governing the localization $\rightarrow$ delocalization transition however turns out to be different in the two cases.

The complexity of many body interactions or external disorder potential associates a degree of uncertainty in the exact determination of the Hamiltonian. As a result, some (or all) elements of the Hamiltonian matrix behave like random variables. The distribution properties of the matrix elements are governed by the nature of intèractions and disorder; their distributions need not be the same, may or may not be correlated, and some of them can be nonrandom, too. For example, a single particle Hamiltonian with a "white noise" disorder potential (that is, no *e-e* interaction or impurity interaction) can be modeled by a random matrix with uncorrelated elements. However the presence of local interactions among impurities or electrons or both will result in a correlated randomization of matrix elements of the Hamiltonian. Unfortunately not much information for random matrices with correlated elements has been available so far. Our objective in this study is to suggest a way to fill in this information gap. Here we show that the eigenvalue distributions of various ensembles, with correlated matrix elements and a multiparametric probability density, appear as nonequilibrium stages of a Brownian type diffusion process. The diffusing eigenvalues evolve with respect to a single parameter which is a function of the correlation strengths between various matrix elements. The parameter is therefore related to complexity of the system represented by the ensemble and can be termed as "complexity" parameter. The solution of the diffusion equation for a given value of complexity parameter gives, therefore, the distribution of eigenvalues, and thereby their correlations, for corresponding system. As discussed later, the diffusion equation can be solved by using its analogy with the one governing the evolution of Brownian ensembles [10]. The analogy also helps in theoretical formulation of the level-density correlations of interacting disordered systems.

The paper is organized as follows. In Sec. II, we derive the diffusion equation for the matrix elements of the Hermitian operators governing the dynamics in complex systems. For a clear exposition of our technique, we first consider the cases with 2nd and 3rd order matrix elements correlations and generalize to *n*th order correlations later on. The diffusion equation for the matrix elements is then used to obtain the equation governing the evolution of the eigenvalues in Sec. III. To maintain the flow of the discussion, only relevant steps are given in Secs. II and III; the details can be found in

the Appendices. As mentioned above, the evolution equation for the eigenvalues turns out to be a very well-known equation and calculation of the eigenvalue correlations from the equation has been discussed in detail many times. We therefore avoid the repetition but give a brief discussion to keep the article self-contained. This is followed, in Sec. IV, by a discussion of the application of our technique to two well-known systems. We conclude in Sec. V by summarizing our main results.

II. SINGLE PARAMETRIC EVOLUTION OF MATRIX ELEMENTS

Let us consider a complex system represented by a $N \times N$ Hermitian matrix H with $H_{kl}=\Sigma_{s=1}^{\beta}(i)^{s-1}H_{kl;s}$ as its matrix elements. The subscript s to a variable refers to one of its components with β as their total number. The parameter β contains the information about underlying symmetry of the system. For example, for systems with time-reversal symmetry and integer angular momentum, the Hamiltonian in a generic representation is a real-symmetric matrix which gives $\beta=1$. The Hamiltonians for a system without time reversal symmetry are, in general, complex Hermitian which gives $\beta=2$. Due to its Hermitian nature, the independent real parameters $H_{kl;s}$, which determine all matrix elements of H, are $\tilde{M}=N+N(N-1)\beta/2$ in number. For notational simplification, let us denote them by H_μ where $\mu \equiv \{kl;s\}$ is a single index which can take a value from $1 \rightarrow M$.

The elements of H describe the overlap between various states of the basis in which H is represented. A complicated nature of interactions or presence of disorder in the system associates a deterministic uncertainty with the matrix elements H_μ. As the interaction between various states is governed by the nature and complexity of the region, the randomness associated with the elements H_μ can be of various types. In general, the complexity of a region can cause multiple interactions between basis states, resulting in correlations between matrix elements. In this section, we consider the cases which can be modeled by an ensemble described by a probability density $\tilde{\rho}(H,b) \propto e^{-F(H)}$ with function $F(H)$ as a sum over various combinations of the matrix elements of H.

A. Correlated Gaussian case

A Gaussian ensemble of Hermitian matrices with correlated elements can be described by a matrix elements distribution

$$\tilde{\rho}(H,b) = C \exp\left[-\sum_{\mu_1=1}^{M} b_{\mu_1} H_{\mu_1} - \sum_{\mu_1,\mu_2=1}^{M} b_{\mu_1\mu_2} H_{\mu_1} H_{\mu_2} \right] = C\rho(H,b), \tag{1}$$

with C as a constant and b as the set of coefficients b_{μ_1} and $b_{\mu_1\mu_2}$. Here the subscripts to a coefficient are indicators of the terms present in the product of which it is a coefficient. Further, in Σ_{μ_1,μ_2}, similar pairs are included only once.

The distribution parameters $b_{2;\mu_1,\mu_2}$ are the measures of correlations between pairs of the matrix elements: $\langle H_{\mu_1} H_{\mu_2} \rangle = \partial \log C / \partial b_{2;\mu_1\mu_2}$. In general, different system conditions can give rise to different sets of distribution parameters b. A slight perturbation of the system due to a change in its parameters perturbs the matrix elements and therefore the probability density $\rho(H,b)$. In the following, we consider a particular flow in the matrix space generated by an operator L (describing the diffusion of the matrix elements with a constant drift and confined by a quadratic potential),

$$L = \sum_{\mu} \frac{\partial}{\partial H_\mu} \left[\frac{g_\mu}{2} \frac{\partial}{\partial H_\mu} + \gamma H_\mu \right], \tag{2}$$

where $g_\mu = 1+\delta_\mu$ with $\delta_\mu = 1$ for $\mu=(kk;s)$ and $\delta_\mu=0$ for $\mu=(kl;s)$, $k \neq l$. The parameter γ is arbitrary, giving the freedom to choose the end of the evolution [10]. The choice of the above form of L is motivated by our desire to obtain the equation governing the eigenvalue-dynamics in a well-known mathematical form.

The evolution of the probability density $\rho(H,b)$, generated by operator L in the matrix space, is related to a multiparametric flow in b space. This can be shown as follows. The probability density being a function of both H and b, the rates of change of ρ with variation of matrix elements and parameters b can be given as

$$\frac{\partial \rho}{\partial H_{\mu_1}} = -\left[b_{\mu_1} + \sum_{\mu_2} \eta_{\mu_1\mu_2}^{(\mu_1)} b_{\mu_1\mu_2} H_{\mu_2} \right] \rho, \tag{3}$$

$$\frac{\partial \rho}{\partial b_{\mu_1\mu_2}} = -H_{\mu_1} H_{\mu_2} \rho, \tag{4}$$

$$\frac{\partial \rho}{\partial b_{\mu_1}} = -H_{\mu_1} \rho, \tag{5}$$

where $\eta_{\mu_1\mu_2}^{(\mu_1)}$ is the frequency of occurrence of the term H_{μ_1} in the combination $H_{\mu_1}H_{\mu_2}$ (i.e., $\eta_{\mu_1\mu_2}^{(\mu_1)}=2$ for $\mu_1=\mu_2$ and 1 for $\mu_1 \neq \mu_2$). With the help of Eqs. (3) and (4), a drift in the matrix space can be written in terms of a drift in parametric space,

$$\sum_{\mu_1} H_{\mu_1} \frac{\partial \rho}{\partial H_{\mu_1}} = -\left[\sum_{\mu_1} c_{\mu_1} H_{\mu_1} \rho + \sum_{\mu_1,\mu_2} c_{\mu_1\mu_2} H_{\mu_2} H_{\mu_1} \right] \rho = \sum_{\mu_1} c_{\mu_1} \frac{\partial \rho}{\partial b_{\mu_1}} + \sum_{\mu_1,\mu_2} c_{\mu_1\mu_2} \frac{\partial \rho}{\partial b_{\mu_1\mu_2}}, \tag{6}$$

where $c_{1;\mu}=b_\mu$ and $c_{\mu_1\mu_2}=\eta_{\mu_1\mu_2}^{(\mu_1)} b_{\mu_1\mu_2}$. Similarly a diffusion in the matrix space can be expressed as a combination of drifts in parametric space,

$$\frac{\partial^2\rho}{\partial H_{\mu_1}^2} = (c_{\mu_1}^2 - c_{\mu_1\mu_1})\rho + 2\sum_{\mu_2} c_{\mu_1}c_{\mu_1\mu_2}H_{\mu_2} + \sum_{\mu_2,\mu_3} c_{\mu_1\mu_2}c_{\mu_1\mu_3}H_{\mu_2}H_{\mu_3}\rho \tag{7}$$

$$= (c_{\mu_1}^2 - c_{\mu_1\mu_1})\rho - 2\sum_{\mu_2} c_{\mu_1}c_{\mu_1\mu_2}\frac{\partial\rho}{\partial b_{\mu_2}} - \sum_{\mu_2,\mu_3} c_{\mu_1\mu_3}c_{\mu_2\mu_3}\frac{\partial\rho}{\partial b_{\mu_2\mu_3}}. \tag{8}$$

A substitution of the above equalities in Eq. (2) gives us the following,

$$L\rho = T\rho + \tilde{C}\rho, \tag{9}$$

with $\tilde{C}=\Sigma_{\mu_1}[\gamma+(g_{\mu_1}/2)(c_{\mu_1}^2-c_{\mu_1\mu_2})]$ and T as the generator of the dynamics in the parametric space,

$$T \equiv \sum_{\mu_1} f_{\mu_1}\frac{\partial}{\partial b_{\mu_1}} + \sum_{\mu_1,\mu_2} f_{\mu_1\mu_2}\frac{\partial}{\partial b_{\mu_1\mu_2}}. \tag{10}$$

Here $f_{\mu_1}=\gamma c_{\mu_1}-\Sigma_{\mu_2}g_{\mu_2}c_{\mu_2}c_{\mu_1\mu_2}$ and $f_{\mu_1\mu_2}=\gamma c_{\mu_1\mu_2}-(1/2)\Sigma_{\mu_3}g_{\mu_3}c_{\mu_1\mu_3}c_{\mu_2\mu_3}$.

The above equation appears complicated, with many parametric derivatives present on its right side. However it is possible to map the multiparametric flow in the M-dimensional b space to a single parametric drift in another parametric space, say y space, consisting of variables y_i, $i=1\rightarrow M$ where $M=[3+\tilde{M}]\tilde{M}/2$. In other words, the generator T of the flow in the b space can be reduced to a partial derivative with respect to just one y-space variable, say y_1:

$$T(y[b])\rho \equiv \left.\frac{\partial\rho}{\partial y_1}\right|_{y_2,\ldots,y_M}. \tag{11}$$

The desired transformation $b\rightarrow y$ required to convert Eq. (10) into the form (11) can be obtained as follows. By using the definition $\partial/\partial x=\Sigma_{k=1}^{M}(\partial y_k/\partial x)(\partial/\partial y_k)$, with x as various b parameters, $T(b)$ [Eq. (10)] can be transformed in terms of the derivatives with respect to y,

$$T\rho = \sum_k A_k\frac{\partial\rho}{\partial y_k}, \tag{12}$$

where

$$A_k \equiv \sum_{\mu_1} f_{\mu_1}\frac{\partial y_k}{\partial b_{\mu_1}} + \sum_{\mu_1,\mu_2} f_{\mu_1\mu_2}\frac{\partial y_k}{\partial b_{\mu_1\mu_2}}. \tag{13}$$

Equation (12) can be reduced in the desired form of Eq. (11), if the transformation $b\rightarrow y$ satisfies the following condition:

$$A_k = \delta_{k1} \quad \text{for } k = 1\rightarrow M. \tag{14}$$

The parameters y as a function of b can now be obtained by solving the set of conditions (14),

$$y_1 = \sum_{\mu_1}\int db_{\mu_1}z_{\mu_1}^{(1)}G + \sum_{\mu_1,\mu_2}\int db_{\mu_1\mu_2}z_{\mu_1\mu_2}^{(1)}G + \text{const}, \tag{15}$$

where $G=[\Sigma_{\mu_1}z_{\mu_1}^{(1)}f_{\mu_1}+\Sigma_{\mu_1,\mu_2}z_{\mu_1\mu_2}^{(1)}f_{\mu_1\mu_2}]^{-1}$ and the set $z^{(1)}$ of the functions $z_{\mu_1}^{(1)}$, $z_{\mu_1\mu_2}^{(1)}$ are chosen such that the ratio

$$\left[\sum_{\mu_1} z_{\mu_1}^{(1)}db_{\mu_1} + \sum_{\mu_1\mu_2} z_{\mu_1\mu_2}^{(1)}db_{\mu_1\mu_2}\right]G \tag{16}$$

is a complete differential (see Appendix B). Note it is possible that the ratio (16) can be made an exact differential for different sets $z^{(1)}$ which will lead to different solutions of y. However, any two such solutions for y are different from each other only by a constant (Appendix B).

The conditions (14) further imply that parameters y_k, $k>1$ behave as the constants of the dynamics (generated by L) in the matrix space,

$$y_k = \sum_{\mu_1}\int db_{\mu_1}z_{\mu_1}^{(k)} + \sum_{\mu_1,\mu_2}\int db_{\mu_1\mu_2}z_{\mu_1\mu_2}^{(k)} + \text{const} \quad \text{for } k>1, \tag{17}$$

with $\Sigma_{\mu_1}z_{\mu_1}^{(k)}f_{\mu_1}+\Sigma_{\mu_1,\mu_2}z_{\mu_1\mu_2}^{(k)}f_{\mu_1\mu_2}=0$ for $k>1$.

The substitution of Eq. (11) in Eq. (9) gives the single parametric evolution of the joint probability density $\rho(H)$ in the matrix space

$$\sum_{\mu}\frac{\partial}{\partial H_\mu}\left[\frac{\partial\tilde{\rho}}{\partial H_\mu} + \gamma H_\mu\tilde{\rho}\right] = \frac{\partial\tilde{\rho}}{\partial y_1}, \tag{18}$$

where y_1 is given by Eq. (15). As the distribution parameters depend on the complexity of the system, y_1 can be termed as the complexity parameter.

The parametric space transformation $b\rightarrow y$ maps the probability density $\rho(H,b)$ to $\rho(H,y(b))$. As a result, ρ depends on various parameters y_k, $k=1\rightarrow\tilde{N}$. However, Eq. (18) implies that the diffusion, generated by the operator L in the matrix space, is governed by y_1 only; the rest of them, namely y_k, $k>1$, remain constant during the evolution. Note it is always possible to define a transformation from the set $b\rightarrow y$ with y_k, $k>1$ as constants of the dynamics generated by L. This can be explained as follows. A matrix element, say H_{ij}, describes how a basis state ψ_i interacts with state ψ_j through H. This results in dependence of the matrix element correlations and, thereby, of the set b, on the basis parameters, e.g., basis indices. As the basis remains fixed during the evolution, the suitable functions of basis parameters can be chosen to play the role of y_k, $k>1$. (Note a similar transformation has been used to obtain a single parametric evolution of multiparametric Gaussian ensembles of Hermitian matrices; see Ref. [10] for details).

B. Non-Gaussian case with third order matrix elements correlations

Let us consider an ensemble of Hermitian matrices with a 3rd correlations among its matrix elements, described by a probability density

$$\tilde{\rho}(H,b) = C\exp\Bigg[-\sum_{\mu_1=1}^{M} b_{\mu_1}H_{\mu_1} - \sum_{\mu_1,\mu_2=1}^{M} b_{\mu_1\mu_2}H_{\mu_1}H_{\mu_2} - \sum_{\mu_1,\mu_2,\mu_3=1}^{M} b_{\mu_1\mu_2\mu_3}H_{\mu_1}H_{\mu_2}H_{\mu_3}\Bigg] = C\rho(H,b). \tag{19}$$

Again the parameters $b_{\mu_1\mu_2\mu_3}$ are the measures of correlations among three matrix elements: $\langle H_{\mu_1}H_{\mu_2}H_{\mu_3}\rangle = \partial \log C/\partial b_{\mu_1\mu_2\mu_3}$. Proceeding as in the Gaussian case, we get

$$\frac{\partial\rho}{\partial H_{\mu_1}} = -\Bigg[c_{\mu_1} + \sum_{\mu_2} c_{\mu_1\mu_2}H_{\mu_2} + \sum_{\mu_2,\mu_3} c_{\mu_1\mu_2\mu_3}H_{\mu_2}H_{\mu_3}\Bigg]\rho, \tag{20}$$

$$\frac{\partial\rho}{\partial b_{\mu_1\mu_2\mu_3}} = -H_{\mu_1}H_{\mu_2}H_{\mu_3}\rho. \tag{21}$$

The derivatives $\partial\rho/\partial b_{\mu}$ and $\partial\rho/\partial b_{\mu_1\mu_2}$ remain the same as in the Gaussian case [given by Eqs. (4) and (5)],

$$\sum_{\mu_1} H_{\mu_1}\frac{\partial\rho}{\partial H_{\mu_1}} = \sum_{\mu_1} c_{\mu_1}\frac{\partial\rho}{\partial b_{\mu_1}} + \sum_{\mu_1,\mu_2} c_{\mu_1\mu_2}\frac{\partial\rho}{\partial b_{\mu_1\mu_2}} + \sum_{\mu_1,\mu_2,\mu_3} c_{\mu_1\mu_2\mu_3}\frac{\partial\rho}{\partial b_{\mu_1\mu_2\mu_3}}, \tag{22}$$

where $c_{\mu_1\mu_2\mu_3} = \eta^{(\mu_1)}_{\mu_1\mu_2\mu_3} b_{\mu_1\mu_2\mu_3}$ with $\eta^{(\mu_1)}_{\mu_1\mu_2\mu_3}$ as the frequency of occurrence of the term H_{μ_1} in the combination $H_{\mu_1}H_{\mu_2}H_{\mu_3}$. The second order derivative of ρ with respect to H_{μ} can be calculated by differentiating Eq. (20) which, on combining with Eq. (21), again leads to the form $L\tilde{\rho} = (T+\tilde{C})\rho$. Here L is still given by Eq. (2); however, the generator T now contains the first order parametric derivatives as well as their products,

$$T\rho = \sum_{\mu_1} f_{\mu_1}\frac{\partial\rho}{\partial b_{\mu_1}} + \sum_{\mu_1,\mu_2} f_{\mu_1\mu_2}\frac{\partial\rho}{\partial b_{\mu_1\mu_2}} + \sum_{\mu_1,\mu_2,\mu_3} f_{\mu_1\mu_2\mu_3}\frac{\partial\rho}{\partial b_{\mu_1\mu_2\mu_3}} + \sum_{\mu_1,\mu_2,\mu_3\mu_4\mu_5} c_{\mu_1\mu_2\mu_3}c_{\mu_1\mu_4\mu_5}\frac{1}{\rho}\frac{\partial\rho}{\partial b_{\mu_2\mu_3}}\frac{\partial\rho}{\partial b_{\mu_4\mu_5}}, \tag{23}$$

with $f_{\mu_1} = \gamma c_{\mu_1} - (1/2)\Sigma_{\mu_2} g_{\mu_2}[2c_{\mu_2}c_{\mu_1\mu_2} + c_{\mu_2\mu_2\mu_1}(\eta^{(\mu_1)}_{\mu_2\mu_2\mu_1} - 1)]$, $f_{\mu_1\mu_2} = \gamma c_{\mu_1\mu_2} - (1/2)\Sigma_{\mu_3} g_{\mu_3}[c_{\mu_1\mu_3}c_{\mu_2\mu_3} + 2c_{\mu_3}c_{\mu_1\mu_2\mu_3}]$ and $f_{\mu_1\mu_2\mu_3} = \gamma c_{\mu_1\mu_2\mu_3} - \Sigma_{\mu_4} g_{\mu_4} c_{\mu_2\mu_4}c_{\mu_1\mu_3\mu_4}$.

Similar to the Gaussian case, Eq. (23) can also be reduced to a single parametric derivative, namely form (11), by using a transformation from the set b to another set y. The operator T in this case transforms as

$$T\rho = \sum_k A_k\frac{\partial\rho}{\partial y_k} + \sum_{i,k} B_{ik}\frac{\partial\rho}{\partial y_i}\frac{\partial \log\rho}{\partial y_k}, \tag{24}$$

where

$$A_k = \sum_{\mu_1} f_{\mu_1}\frac{\partial y_k}{\partial b_{\mu_1}} + \sum_{\mu_1,\mu_2} f_{\mu_1\mu_2}\frac{\partial y_k}{\partial b_{\mu_1\mu_2}} + \sum_{\mu_1,\mu_2,\mu_3} f_{\mu_1\mu_2\mu_3}\frac{\partial y_k}{\partial b_{\mu_1\mu_2\mu_3}}, \tag{25}$$

$$B_{ik} = \sum_{\mu_1,\mu_2,\mu_3\mu_4\mu_5} c_{\mu_1\mu_2\mu_3}c_{\mu_1\mu_4\mu_5}\frac{\partial y_i}{\partial b_{\mu_2\mu_3}}\frac{\partial y_k}{\partial b_{\mu_4\mu_5}}. \tag{26}$$

The reduction of Eq. (24) for $T\rho$ to a single derivative $\partial\rho/\partial y_1$ will impose the following conditions on the transformation $b \to y$:

$$A_k = N\delta_{k1}, \tag{27}$$

$$B_{ik} + B_{ki} = 0. \tag{28}$$

By solving conditions (27), y_1 can be obtained as a function of the parameters b,

$$y_1 = \sum_{\mu_1}\int db_{\mu_1} z^{(1)}_{\mu_1} G + \sum_{\mu_1,\mu_2}\int db_{\mu_1\mu_2} z^{(1)}_{\mu_1\mu_2} G + \sum_{\mu_1,\mu_2,\mu_3}\int db_{\mu_1\mu_2\mu_3} z^{(1)}_{\mu_1\mu_2\mu_3} G + \text{const}, \tag{29}$$

where

$$G = \Big[\sum_{\mu_1} z^{(1)}_{\mu_1} f_{\mu_1} + \sum_{\mu_1,\mu_2} z^{(1)}_{\mu_1\mu_2} f_{\mu_1\mu_2} + \sum_{\mu_1\mu_2\mu_3} z^{(1)}_{\mu_1\mu_2\mu_3} f_{\mu_1\mu_2\mu_3}\Big]^{-1}.$$

Again the choice of the functions $z^{(1)}$ is so as to make the ratio

$$\Bigg[\sum_{\mu_1} z^{(1)}_{\mu_1} db_{\mu_1} + \sum_{\mu_1\mu_2} z^{(1)}_{\mu_1\mu_2} db_{\mu_1\mu_2} + \sum_{\mu_1\mu_2\mu_3} z^{(1)}_{\mu_1\mu_2\mu_3} db_{\mu_1\mu_2\mu_3}\Bigg] G \tag{30}$$

a complete differential. However, the set $z^{(1)}$ in this case has to satisfy an extra set of conditions given by Eq. (29). Again as in the Gaussian case, the parameters y_j, $j>1$ behave as constants of dynamics for this case, too,

$$y_k = \sum_{\mu_1}\int db_{\mu_1} z^{(k)}_{\mu_1} + \sum_{\mu_1,\mu_2}\int db_{\mu_1\mu_2} z^{(k)}_{\mu_1\mu_2} + \sum_{\mu_1,\mu_2\mu_3}\int db_{\mu_1\mu_2} z^{(k)}_{\mu_1\mu_2\mu_3} \quad \text{for } k>1, \tag{31}$$

with arbitrarily chosen functions $z^{(k)}_{\mu}$ satisfying the constraint $\Sigma_{\mu_1} z^{(k)}_{\mu_1} f_{\mu_1} + \Sigma_{\mu_1,\mu_2} z^{(k)}_{\mu_1\mu_2} f_{\mu_1\mu_2} + \Sigma_{\mu_1,\mu_2} z^{(k)}_{\mu_1\mu_2\mu_3} f_{\mu_1\mu_2\mu_3} = 0$ for $k>1$ as well as the conditions given by Eq. (28).

Using the $b \to y$ transformation given by Eqs. (29) and (31), the dynamics of $\rho(H)$ with third order matrix element correlations can again be described by Eq. (18). Note that although the evolution is governed by the same equation, the complexity parameters are different for Gaussian and non-Gaussian cases.

C. General case

A generalized ensemble of Hermitian matrices with correlated elements can be described by a matrix elements distribution $\tilde{\rho}(H)=C\rho(H)$ where

$$\rho(H)=\prod_{r=1}^{n}\exp\left[-\sum_{p(r)}b_{p(r)}\left(\prod_{j=1}^{r}H_{\mu_{P_j}}\right)\right], \tag{32}$$

with C as a normalization constant. Here symbol $p(r)$ refers to a combination of r elements chosen from a set of total $\tilde{M}=N+N(N-1)\beta/2$ elements of upper (or lower) diagonal matrix; note the terms present in a given combination need not be all different. The $\Pi_{j=1}^{r}H_{\mu_{P_j}}$ implies a product over r terms present in the pth combination with coefficient $b_{p(r)}$ as a measure of their correlation: $\langle\Pi_{j=1}^{r}H_{\mu_{P_j}}\rangle=\partial\log C/\partial b_{p(r)}$. The $\Sigma_{p(r)}$ is a sum over all possible combinations [total $(\tilde{M})^r$] of r elements chosen from a set of total $\tilde{M}$ of them; the sum includes only different combinations. [Henceforth subscript $p(r)$ will be written as p only unless details are required for clarification.]

Using Eq. (32), the partial derivatives of ρ in matrix space and parametric space can be given as

$$\frac{\partial\rho}{\partial H_\mu}=-\sum_{r=1}^{n}\sum_{p}\left[b_p\left(\prod_{j=1}^{r}H_{\mu_{p_j}}\right)\sum_{j=1}^{r}\frac{\partial\log H_{\mu_{p_j}}}{\partial H_\mu}\right]\rho, \tag{33}$$

$$\frac{\partial\rho}{\partial b_{p(r)}}=-\left(\prod_{j=1}^{r}H_{\mu_{P_j}}\right)\rho. \tag{34}$$

As a result, a drift in the matrix space can be related to a drift in the parametric space:

$$\sum_{\mu}H_\mu\frac{\partial\rho}{\partial H_\mu}=\sum_{\mu}\sum_{r,p(r)}\eta_p^{(\mu)}b_p\frac{\partial\rho}{\partial b_p}, \tag{35}$$

where the term $\eta_{p(r)}^{(\mu)}=\Sigma_{m=1}^{r}\delta(\mu_{p_m};\mu)$ counts the frequency of occurrence of the element H_μ in the pth combination of r elements [$\delta(\mu_{p_j},\mu)=1$ if $\mu_{p_j}=\mu$ and 0 if $\mu_{p_j}\neq\mu$]. Similarly, a diffusion in the matrix space is related to a nonlinear flow in the parametric space,

$$\frac{\partial^2\rho}{\partial H_\mu^2}=\sum_{r=1}^{n}\sum_{p(r)}\eta_p^{(\mu)}b_p\left[\sum_{r'>(n+2-r)}\sum_{p'(r')}\eta_{p'}^{(\mu)}b_{p'}\frac{\partial\rho}{\partial b_{p+\mu}}\frac{\partial\rho}{\partial b_{p'+\mu}}\rho^{-1}-\sum_{r'\leq(n+2-r)}\sum_{p'(r')}\eta_{p';\mu}b_{p'}\frac{\partial\rho}{\partial b_{p'+p-\mu^2}}+(\eta_p^{(\mu)}-1)\frac{\partial\rho}{\partial b_{p-\mu^2}}\right]. \tag{36}$$

Here the notation $A+B$ refers to a combination which contains the elements of both A and B. Similarly $A-B$ indicates dropping of all the elements of B from A. Further the notation μ^k is used to denote a combination H_μ^k (that is, kth power of H_μ).

Again, the matrix space flow generated by the operator L [Eq. (2)] can be related to a parametric flow generated by the operator T, $L\rho=T\rho+\tilde{C}\rho$, where T is now given by

$$T\rho\equiv\sum_{\mu}\frac{g_\mu}{2}\sum_{r=1}^{n}\sum_{p(r)}\left[\sum_{r'>(n+2-r)}^{m}\sum_{p'(r')}h_{pp'}\frac{\partial\log\rho}{\partial b_{p'}}+f_p\right]\frac{\partial\rho}{\partial b_p}, \tag{37}$$

with

$$h_{pp'}=b_ub_{u'}\left[\sum_{\mu}g_\mu\eta_u^{(\mu)}\eta_{u'}^{(\mu)}\right]$$

and

$$f_p=-\sum\nolimits_{r',p'(r')}b_{p'}b_q\left[\sum_{\mu}g_\mu\eta_{p'}^{(\mu)}\eta_q^{(\mu)}\right]+\left[\sum_{\mu}\eta_v^{(\mu)}g_\mu(\eta_v^{(\mu)}-1)\right]b_v+2\gamma\left[\sum_{\mu}\eta_p^{(\mu)}\right]b_p.$$

Here q refers to a combination of $r_1=r'-r+2$ elements, such that $q(r_1)\equiv p'(r')-p(r)+(kl)^2$. Similarly u,u',v refer to combinations $u(r_0)=p(r)+1$, $u(r_1)=p'(r')+1$ and $v(r_2)=p(r)+2$ of $r_0=r+1$, $r_1=r+1$, and $r_2=r+2$ elements, respectively.

The desired transformation $b\to y$ required to convert Eq. (37) into form $T(y[b])\rho\equiv\partial\rho/\partial y_1$ can be obtained as follows. The substitution of $\partial/\partial b_{p(r)}=\Sigma_{k=1}^{M}D_{kj}(\partial/\partial y_k)$ in Eq. (37) with $D_{kj}\equiv\partial y_k/\partial b_{p(r)}$ transforms T in the form of Eq. (24) where A_k and B_{ik} are now given as

$$A_k(y,b)=\sum_{r}\sum_{p(r)}f_pD_{kp}, \tag{38}$$

$$B_{ik}(y,b)=\sum_{r,p(r)}\sum_{r',p'(r')}h_{pp'}(x)D_{ip}D_{kp'}. \tag{39}$$

Again, for desired reduction of T to $\partial\rho/\partial y_1$, the transformation $b\to y$ should satisfy conditions given by Eqs. (27) and (28). The conditions (27) can then be solved to obtain the variables in set y as functions of variable in set b (see Appendix B)

$$y_k=\sum_{r,p(r)}\int db_pz_p^{(k)}[1-\delta_{k1}+G_k]+\text{const}, \tag{40}$$

with $G_k=\delta_{k1}[\Sigma_{r,p(r)}z_p^{(1)}f_p]^{-1}$ and $z_p^{(k)}$ as arbitrary functions which make $\Sigma_{r,p(r)}db_pz_p^{(k)}[1+(G-1)\delta_{k1}]$ an exact differential and satisfy the constraint $\Sigma_{r,r'}\Sigma_{p(r),p'(r')}z_p^{(i)}z_{p'}^{(k)}f_{pp'}G_1^2=0$ [the latter is required by the conditions (39)].

We find, therefore, that the diffusion of probability density for ensembles of Hermitian operators with correlated matrix elements (any order), is governed by a single parameter y_1 with evolution described by Eq. (18). The eigenvalue statistics of the above ensembles can therefore be studied by an exact diagonalization of Eq. (18).

III. SINGLE PARAMETRIC EVOLUTION OF EIGENVALUES

The eigenvalue equation for the matrix H can be given as $HU=\underline{\lambda}U$ with $\underline{\lambda}$ as the diagonal matrix with eigenvalues λ_i of H as its matrix elements and U as the eigenvector

matrix (unitary for complex Hermitian case and orthogonal for real-symmetric case). The probability density $P(\underline{E},y,b)$ of finding eigenvalues λ_i between E_i and E_i+dE_i at a given Y can then be obtained from the matrix elements distribution,

$$P(\underline{E},y[b]) = \int \prod_{i=1}^{N} \delta(E_i - \lambda_i)\tilde{\rho}(H,y[b])dH. \tag{41}$$

Here $\underline{E}$ refers to a diagonal matrix with elements $E_1,\ldots,E_N$. As the $Y \equiv y_1$ dependence of P in Eq. (41) enters only through ρ, a derivative of P with respect to Y can be written as follows:

$$\frac{\partial P}{\partial Y} = \int \prod_{i=1}^{N} \delta(E_i - \lambda_i)\frac{\partial \tilde{\rho}}{\partial Y}dH. \tag{42}$$

The diffusion equation for $\tilde{\rho}$, namely Eq. (18), can now be used to rewrite Eq. (42) as

$$\frac{\partial P}{\partial Y} = I_1 + I_2, \tag{43}$$

where

$$I_1 = \gamma \sum_{\mu} \int \delta(E-\lambda)\frac{\partial(H_\mu \tilde{\rho})}{\partial H_\mu}dH, \tag{44}$$

$$I_2 = \sum_{\mu} \int \prod_{i=1}^{N} \delta(E_i - \lambda_i)\frac{\partial^2 \tilde{\rho}}{\partial H_\mu^2}dH. \tag{45}$$

The integral I_1 can be simplified by using integration by parts

$$I_1 = -\gamma \sum_{\mu} \int \left[\frac{\partial}{\partial H_\mu}\prod_{i=1}^{N}\delta(E_i-\lambda_i)\right] H_\mu \tilde{\rho}dH \tag{46}$$

$$= \gamma \sum_{n=1}^{N} \frac{\partial}{\partial E_n}\int \prod_{i=1}^{N}\delta(E_i-\lambda_i)\left[\sum_{\mu}\frac{\partial \lambda_n}{\partial H_\mu}H_\mu\right]\tilde{\rho}dH. \tag{47}$$

A further simplification of the above equation requires a knowledge of the rate of change of eigenvalues of H due to a small change in its matrix elements. The rate can be obtained by using the eigenvalue equation for matrix H along with the unitary (or orthogonal for real-symmetric H) nature of its eigenvectors (see Appendix A). Using Eq. (A2) of Appendix A in Eq. (47) we get

$$I_1 = \gamma \sum_{n} \frac{\partial}{\partial E_n}(E_n P). \tag{48}$$

The second term can similarly be rewritten as follows:

$$I_2 = \sum_{n} \frac{\partial}{\partial E_n}\sum_{\mu}\frac{g_\mu}{2}\int \frac{\partial}{\partial H_\mu}\left(\prod_i \delta(\mu_i - \lambda_i)\frac{\partial \lambda_n}{\partial H_\mu}\right)\tilde{\rho}dH \tag{49}$$

$$= \sum_{m,n}\frac{\partial}{\partial E_n E_m}\int \prod_i \delta(E_i-\lambda_i)\left[\sum_\mu \frac{g_\mu}{2}\frac{\partial \lambda_m}{\partial H_\mu}\frac{\partial \lambda_n}{\partial H_\mu}\right]\tilde{\rho}dH - \sum_m \frac{\partial}{\partial E_n}\int \prod_i \delta(E_i-\lambda_i)\left[\sum_\mu \frac{g_\mu}{2}\frac{\partial^2 \lambda_n}{\partial H_\mu^2}\right]\tilde{\rho}dH. \tag{50}$$

Now by using Eqs. (A3) and (A5) of Appendix A, I_2 can be expressed in terms of eigenvalue derivatives of ρ,

$$I_2 = \sum_n \frac{\partial^2 P}{\partial E_n^2} + \sum_n \frac{\partial}{\partial E_n}\left[\sum_{m\neq n}\frac{\beta P}{E_m - E_n}\right]. \tag{51}$$

A substitution of I_1 and I_2, given by Eqs. (47) and (51), in Eq. (43) leads an equation describing the single parametric evolution of the eigenvalues of ensemble $\rho(H)$,

$$\frac{\partial P}{\partial Y} = \sum_n \frac{\partial}{\partial E_n}\left[\frac{\partial}{\partial E_n} + \sum_{m\neq n}\frac{\beta}{E_m - E_n} + \gamma E_n\right]P. \tag{52}$$

Equation (52) describes the evolution of the eigenvalue density $P(\underline{E},y[b]) \equiv P(\underline{E},Y,y_2,\ldots,y_M)$ due to variation of the parameter Y from an arbitrary initial state, say $P(\underline{E}_0,y[b_0]) \equiv P(\underline{E}_0,Y_0,y_2,\ldots,y_M)$ occurring at $Y=Y_0$. Note here that the parameters y_j $(j>1)$, being constants of motion, have the same value for both initial ensemble $\rho(H_0,b_0)$ as well as $\rho(H,b)$. The evolution of the eigenvalues tends to a steady state in limit $\partial P/\partial Y \to 0$ or $Y \to \infty$. The solution of Eq. (52) in the limit is a Wigner-Dyson ensemble [7]: $P(\underline{E}) = \Pi_{i<j}|E_i - E_j|^\beta e^{-(\gamma/2)\Sigma_k E_k^2}$ (thus a GOE for $\beta=1$ and a GUE for $\beta=2$). Note, under certain conditions, the steady state solution may also correspond to the eigenvalue distribution of an ensemble of antisymmetric Hermitian matrices. A knowledge of the solution of Eq. (52) can now help us in determining the n-level density correlations R_n, defined as $R_n(E_1,E_2,\ldots,E_n;Y) = [N!/(N-n)!]\int \Pi_{j=n+1}^{N} dE_j P(\underline{E};Y - Y_0)$. The first order correlation R_1 is also known as mean level density and its inverse gives the mean level spacing Δ of the full spectrum (that is, average of spacings in full length of the spectrum). By a direct integration, Eq. (52) can also be used to study the evolution of R_n with changing complexity of the system.

Equation (52) is applicable for arbitrary values of the coefficients b; it is therefore valid for the case of uncorrelated Gaussian ensembles too. The latter have been shown to be good models for noninteracting systems [7]. Within random matrix framework, therefore, we find that the energy levels of both systems, interacting as well as noninteracting, undergo a same diffusion process with changing system parameters. As a result, the level statistics (and related physical properties) in the two cases can be described by the same mathematical formulation. However note that, due to different complexity parameters in general, the rate of evolution is different in the two cases.

The advantages of a single parametric formulation of the evolution of eigenvalues is manifold and have been discussed in detail in Ref. [10] in the context of noninteracting systems (or uncorrelated ensembles); the similarity of Eq. (52) with that of Eq. (17) of Ref. [10] allows us to use the discussion given in Secs. I D, I E, and II of Ref. [10] for the correlated ensembles too (with μ replaced by E). However, for completeness sake, we briefly review it here again. Equation (52) is similar to the equation governing the evolution of eigenvalues in the Dyson's Brownian motion model which was originally introduced by Dyson to describe the eigenvalue-dynamics of an ensemble of Hermitian matrices subjected to random perturbation; the nonequilibrium states of this model are known as Brownian ensembles (see Chap. 8 of Ref. [5]). Later on it was shown that when an ensemble H_0 (fixed or random) is subjected to a random perturbation, of strength $\sqrt{Y-Y_0}$, by a standard random matrix ensemble V (described by a probability density $e^{-(\gamma/2)\mathrm{Tr}\,V^2}$), the resulting ensemble $H=H_0+(\sqrt{Y-Y_0})V$ behaves like a Brownian ensemble (BE) (see Chap. 14 of Ref. [5], Chap. 6 of Refs. [6,12] and [10]). Here H_0 and V may belong to a same symmetry class, with $\sqrt{Y-Y_0}$ governing the parametric eigenvalue dynamics, or different symmetry classes with $\sqrt{Y-Y_0}$ as a parameter for symmetry admixing transitions. The statistical properties of BE depend only on the parameter $\sqrt{Y-Y_0}$ besides underlying symmetry and many of their correlations are already known [12].

As discussed in Ref. [12], the mean level density R_1 of a BE changes from an initial state to a semicircular form (typical of Wigner-Dyson ensembles) at the scale of $\gamma(Y-Y_0) \simeq N\Delta_l^2$ with $\Delta_l(E,Y)$ as the local mean level spacing at energy E; its evolution can therefore be described in terms of the parameter $(Y-Y_0)$. However, the transition of level-density correlations to equilibrium, with $(Y-Y_0)$ as the evolution parameter, is rapid, discontinuous for infinite dimensions of matrices [12]. For small Y and large N, a smooth crossover can be seen in terms of a rescaled parameter $\Lambda(E)$:

$$\Lambda(E) = \gamma|Y - Y_0|/\Delta_l^2. \tag{53}$$

The limits $\Lambda \to 0$ and $\Lambda \to \infty$ correspond to the level statistics approaching the initial state and Wigner-Dyson ensembles, respectively. As obvious from the definition of Λ, an intermediate state between two limits occurs when the perturbation $\sqrt{Y-Y_0}$ mixes levels in a finite energy range of r local mean level spacings: $r \simeq \sqrt{Y-Y_0}/\Delta_\eta$, $0<r<N$. For finite size BE, Λ varies smoothly with changing $Y-Y_0$ which results in a continuous family of BEs, parametrized by Λ. However, the level statistics for the large BE (size $N \to \infty$) can be divided into three regions.

(i) Initial regime: $(Y-Y_0)\Delta_l^{-2} \to f(N^{-1})$. If the local mean level spacing Δ_l increases with size N at a rate faster than that of $\sqrt{Y-Y_0}$, the perturbation will mix fewer number of levels as system size increases. The level statistics therefore approaches its initial state in the infinite size limit.

(ii) WD regime: $(Y-Y_0)\Delta_l^{-2} \to f(N)$. Due to change in Δ_η with size N being slower than that of $\sqrt{Y-Y_0}$), even a small change in complexity parameter in this case is capable of mixing the levels in an increasing energy range of many local mean level spacings. This results in an increasing delocalization of eigenfunctions and Wigner-Dyson behavior of level statistics.

(iii) Critical regime. $(Y-Y_0)\Delta_l^{-2} = f(N^0) = \alpha =$ a const. The perturbation in this case mixes only a finite (nonzero), fixed number of levels even when the system is growing in size. The finite, nonzero Λ value in limit $N \to \infty$ therefore gives rise to a third statistics, intermediate between initial ensemble and Wigner-Dyson ensemble, which is known as the critical Brownian ensemble (CBE). This being the case for arbitrary values of α (nonzero and finite), an infinite family of critical BE, characterized by α, can occur during transition from initial ensemble to Wigner-Dyson ensembles.

The same evolution equations of P for correlated ensembles and BE imply a similarity in their eigenvalue distributions for all Y values, under similar initial conditions [that is, $P(E,Y_0)$ same for both the cases]. As a result, one obtains the analogous evolution equations for their correlations R_n, too [see Eq. (16) of Ref. [12]]. The mean level density R_1 of a correlated ensemble can therefore be given by the mean level density of a BE with same $(Y-Y_0)$ value and belonging to a same symmetry class (as that of correlated ensemble). Further the analogy of evolutions of higher order correlations $(n>1)$ in the two cases implies (i) the discontinuity of transition of R_n for infinite size correlated ensembles and (ii) a smooth crossover of R_n for finite correlated ensembles. The crossover parameter for correlated ensembles can again be defined by Eq. (54) where now $Y-Y_0$ is the complexity parameter of the correlated ensemble and Δ_l is its local mean level spacing. Note, in the case of d-dimensional disordered systems of linear size L, the number of states in a volume of linear dimension ζ in d dimensions is $n(0)\zeta^d$ with $n(0)$ as the density of states at Fermi level and ζ as the localization length. Consequently, the typical energy separation between such states is $\Delta_l(E,Y) = (n(0)\zeta^d)^{-1}$. Similarly the mean level spacing of states in the full length of the spectrum is $\Delta(E,Y) = (n(0)L^d)^{-1}$. For disordered systems, the local mean level spacing Δ_l can therefore be expressed in terms of the mean level density R_1 as $\Delta_l = (L/\zeta)^d R_1^{-1}$.

The parameter Λ, being a function of the distribution parameters of the matrix elements, is sensitive to the changes in the system parameters, due to their influence on the system interactions and their uncertainties. Some examples of such system parameters are disorder, dimensionality, boundary and topological conditions, system size, etc. The presence of disorder randomizes the interactions in the system with degree of disorder affecting the distribution parameters b and consequently Λ. The dependence of Λ on dimensionality and boundary conditions can be explained by using a simple example. Consider a $N \times N$ lattice with a Gaussian site disorder as well as a Gaussian type random interaction between nearest-neighbor sites. The lattice Hamiltonian H, in site representation, is a sparse matrix with only $(Z+1)N$ nonzero, independently distributed matrix elements; here Z is the number of nearest neighbors of a site. Consequently only $(Z+1)N$ b parameters (out of N^2) contribute to Y. As the coordination number Z is different for different dimensions and boundaries of the lattice, the Z dependence of Y results in its dependence on the dimensionality as well as boundary

conditions of the system. Further the local mean level spacing is also sensitive to the dimensionality as well as the boundary conditions. A variation of any of the latter parameters can affect both Δ_l and Y and therefore Λ; (see also Ref. [11] where the dependence of Λ on system parameters is explained by considering an example of Anderson Hamiltonian).

The size dependence of Λ also plays a crucial role in determining the level statistics of the correlated ensemble in the crossover regime. In general, both $Y-Y_0$ as well as the local mean level density are the functions of system size N which results in N dependence of Λ. As a consequence, the level statistics in finite systems smoothly approaches one of the two end points, namely, $\Lambda\rightarrow 0$ or $\Lambda\rightarrow\infty$, with increasing system size. However, as in the BE case, the variation of Λ in infinite correlated ensembles may lead to an abrupt transition, with its critical point given by the condition $\Lambda=$size independent. As in the BE case, the size independence of Λ at the critical point results in a level statistics different from the two end points. Note if the size dependence of Δ_l^2 in a correlated ensemble remains different from that of $Y-Y_0$ under all complexity conditions, the system will never undergo a transition in level statistics.

IV. EXAMPLES

In this section, we consider two examples corresponding to 2nd and 3rd order matrix elements correlations and provide the theoretical formulation for 2-point eigenvalue correlations for the cases by using the Brownian ensemble analogy.

A. Quantum Hall system

Let $H=H_0+V(\mathbf{r})$ be the single particle Hamiltonian for a disordered quantum Hall system with H_0 as the kinetic energy of the electrons and $V(\mathbf{r})$ as a space correlated disordered potential, e.g., $\langle V(\mathbf{r})V(\mathbf{r}')\rangle=f(\mathbf{r},\mathbf{r}')$ (with $\langle V(\mathbf{r})\rangle=0$). Using the Landau states $\psi_{nk}(r)\equiv\langle\mathbf{r}|nk\rangle$ (the eigenstates of H_0) as the basis, H can be written as $H_{nk;n'k'}=\epsilon_n\delta_{n,n'}\delta_{k,k'}+V_{nk;n'k'}$ where $H_{nk;n'k'}\equiv\langle nk|H|n'k'\rangle$ and $\epsilon_n=(n+1/2)\hbar\omega$ as the eigenvalues of H_0. The interaction between impurities results in a correlation of the matrix elements of V and thereby H [13]:

$$\langle V_{n_1k_1;n_2k_2}V_{n_3k_3;n_4k_4}\rangle = \int d\mathbf{r}d\mathbf{r}'\psi^*_{n_1k_1}(\mathbf{r})\psi_{n_2k_2}(\mathbf{r})\psi^*_{n_3k_3}(\mathbf{r}')\psi_{n_4k_4}(\mathbf{r}')f(\mathbf{r},\mathbf{r}'). \tag{54}$$

The parameter Λ can now be determined if mean level spacing Δ and the real-space correlations for the potential V are explicitly known. For example, consider the case when magnetic field B becomes much stronger than the disorder potential. The Hamiltonian matrix H in this case is divided into various independent blocks (each corresponding to a different Landau level) and the statistics of energy states in each Landau level can be discussed independently [13]. For a Gaussian type disorder $\langle V(\mathbf{r})V(\mathbf{r}')\rangle=(V_0^2/2\pi\sigma^2)e^{-|\mathbf{r}-\mathbf{r}'|^2/2\sigma^2}$, the matrix element correlations in the lowest Landau level $n=0$ can be given as $\langle V_{0i;0j}V_{0k;0l}\rangle\equiv\langle V_{ij}V_{kl}\rangle$, where

$$\langle V_{ij}V_{kl}\rangle=(V_0^2/l_cL_y\alpha\sqrt{2\pi})\delta(i-j,l-k)e^{-(i-j)^2\alpha^2/2}e^{-(i-k)^2/2\alpha^2}, \tag{55}$$

with $\alpha^2=(1+\sigma^2/l_c^2)$ as a measure of the correlation length of the potential relative to the magnetic length $l_c=(\hbar/eB)^{1/2}$ [13]. Using the notation $H_{0k;0l}\equiv H_{kl}$, the distribution parameters of the matrix elements of the Hamiltonian in the Landau level $n=0$ can be given as

$$\langle H_{ij}\rangle=\epsilon_0\delta_{ij}, \tag{56}$$

$$\langle H_{ij;s}H_{kl;s}\rangle=\epsilon_0^2\delta_{ij}\delta_{kl}+\langle V_{ij;s}V_{kl;s}\rangle, \tag{57}$$

with $\langle V_{ij;s}V_{kl;s}\rangle$ as the correlations between different components of the elements of V,

$$\langle V_{ij;s}V_{kl;s'}\rangle=\frac{1}{2}[\langle V_{ij}V_{lk}\rangle+(-1)^{s-1}\langle V_{ij}V_{kl}\rangle]\delta_{ss'}. \tag{58}$$

The distribution of the local Hamiltonian for the lowest Landau level can then be represented by Eq. (1) with parameters b obtained from Eqs. (56) and (57) [here $\mu_1\equiv(ij;s)$ and $\mu_2\equiv(kl;s)$]; see Appendix C for an example. A substitution of the b parameters in Eq. (15) gives us the complexity parameter governing the energy level dynamics in the lowest Landau level. As shown in Appendix C by a simple case $N=2$, the parameters b and, therefore Y, turn out to be a function of the disordered potential α, V_0, system length L_y as well as magnetic field B and can be varied by changing any one of them.

The above discussion is valid for higher order Landau levels too, with the complexity parameter still described by Eq. (15); however, the coefficients $b_{\mu_1\mu_2}$ are different for different Landau levels (see Ref. [13] for the matrix element correlations of potential V). The rate of transition of level statistics therefore differs, in general, from one Landau level to another. For weak magnetic fields, where various Landau levels cannot be considered as independent, H can still be represented by the ensemble Eq. (1). However, now the number of coefficients $b_{\mu_1\mu_2}$ which contribute to Y is much larger (due to correlations between levels in two different Landau levels).

In the absence of disorder, under the independent Landau level approximation, all energy levels in a given Landau level are degenerate and matrix $H=H_0$ is a diagonal matrix with a Poisson behavior for its eigenvalues (due to dominance of zero spacings). The switching of disorder removes the degeneracy and delocalizes the wave function if the impurities are interacting. The degree of delocalization depends on the strength of impurity interactions with respect to the magnetic field strength B. If the latter is strong enough to mix the levels in an energy range of many mean level spacings (which corresponds to the limit $H\approx V$), the energy levels of H show a GUE behavior. (This is similar to the case of strongly interacting many body systems, e.g., the statistics of

resonances in complex nuclei which can be well modeled by GOE or GUE [7].) Under an intermediate state of disorder, therefore, the ensemble H lies between the Poisson ensemble and GUE and can be modeled by Eq. (1). The level statistics for this case can then be given by the one for a BE appearing during a *Poisson*→GUE transition. The two point correlator R_2 [12] for states in the lowest Landau level can therefore be given as

$$R_2(r;\Lambda) - R_2(r;\infty) = \frac{4}{\pi}\int_0^\infty dx \int_{-1}^1 dz \cos(2\pi r x)\exp[-8\pi^2\Lambda x(1+x+2z\sqrt{x})] \times\left(\frac{\sqrt{(1-z^2)}(1+2z\sqrt{x})}{1+x+2z\sqrt{x}}\right), \tag{59}$$

where $R_2(r,\infty)=1-\sin^2(\pi r)/\pi^2 r^2$ (the GUE limit); the above formulation was obtained for the BE in Ref. [12].

Let us now consider the case with weak magnetic fields where the interaction between various Landau levels cannot be ignored. The eigenvalue spectrum of H_0 for this case behaves as a uniform spectrum [13] (that is, an initial spectrum of uniform spacing). The switching of interacting-impurities potential V again results in broadening of the wave functions. In the limit where impurity interactions are strong enough to mix energy levels in different Landau levels (that is $H\approx V$), the eigenvalues of H show a GUE behavior. The varying degree of the interaction between impurities therefore leads to a transition of the level statistics from uniform spectrum→GUE behavior. The two point correlation for the level statistics at any intermediate stage of impurity interaction can then be given by that of a BE appearing during the uniform spectrum→GUE transition:

$$R_2(r;\Lambda) - R_2(r;\infty) = 2\sum_{q=-\infty}^{\infty} e^{-8\pi^2 q^2\Lambda}\int_0^1 dx(1-x)e^{-8\pi^2 qx\Lambda} \times[\cos(2\pi qr)-\cos(2\pi(q+x)r)], \tag{60}$$

with $\Lambda\to\infty$ corresponding to GUE limit.

B. Disordered systems with fermionic interactions

Consider a general Hamiltonian for spinless interacting fermions

$$H=\sum_{ij} V_{ij}a_i^\dagger a_j + \frac{1}{4}\sum_{ijkl} U_{ijkl}a_i^\dagger a_j^\dagger a_l a_k. \tag{61}$$

[Here the states $|i\rangle=a_i^\dagger|0\rangle$ describe a fixed basis of m single-particle states with V_{ij} as matrix elements of the one body Hamiltonian and U_{ijkl} as the antisymmetrized matrix elements of the two body interaction U. The presence of disorder randomizes both V and U. For example, for V as a white noise, its matrix elements are independently distributed random variables. However the fermionic interaction results in correlations among matrix elements of U:

$$\langle U_{ijkl}U_{klmn}U_{mnij}\rangle = \int d^3r_1 d^3r_2 d^3r_3 \psi_{ij}^*(r_1)\psi_{kl}(r_1)\psi_{kl}^*(r_2) \times\psi_{mn}(r_2)\psi_{mn}^*(r_3)\psi_{ij}(r_1)f(r_1,r_2)f(r_2,r_3)f(r_3,r_1), \tag{62}$$

with $f(\mathbf{r}_1,\mathbf{r}_2)$ as the interaction between two fermions at positions $\mathbf{r}_1$ and $\mathbf{r}_2$. The Hamiltonian H will therefore be a matrix with varying degree of correlations between its elements and can be represented by ensemble Eq. (19). In absence of fermionic interaction, the matrix $H=V$ and its statistical properties depend on the degree of disorder. For example, in the presence of strong disorder, the eigenvalues of V show a Poisson distribution with localized eigenfunctions [11,3,7]. The weak disorder limit of V shows a Wigner-Dyson distribution for its eigenvalues with extended eigenfunctions. Similarly in absence of disorder, $H=U$ and its statistical behavior is governed by the fermionic density in various parts of the system. The presence of almost uniform fermionic density in the system leads to delocalization of eigenfunctions and U behaves like an ensemble of antisymmetric Hermitian matrices (see Ref. [5] for details on antisymmetric random matrices). However nonuniform electronic density in various parts of the system (that is, stronger interactions in certain parts of the system as compared to others) can result in localization of wave functions and thereby a Poisson behavior for the eigenvalues of U.

In the presence of both disorder as well as fermionic interaction, the behavior of H is governed by the intercompetition between them. In this case, it is preferable to represent H in the $N=m^2$ dimensional basis of two particle states $|ij\rangle$; the choice of the basis results in appearance of the disorder elements as the diagonal elements of H and fermionic interaction elements as the off-diagonals: $\langle ij|H|ij\rangle=V_{ij}$ and $\langle ij|H|kl\rangle=U_{ijkl}$. The changing strength of the fermionic interaction in the presence of disordered potential subjects the level statistics to undergo a transition from the initial state (given by the statistics of V) to Wigner-Dyson statistics (when $U\simeq V$). However, in the limit when the disorder potential becomes negligible as compared to fermionic interactions, the level statistics of H approaches that of antisymmetric matrices. As obvious, the level statistics for all other cases, corresponding to different strengths of disorder potential and fermionic density, will lie on the transition curve from Poison→Wigner-Dyson→antisymmetric ensembles. Using Eq. (19) as the model for intermediate states, the level statistics for this case can then be described by BE lying between Poison→antisymmetric ensemble.

The ground state properties of electrons in nanoparticles or quantum dots, that is, finite systems of fermions interacting via Coulomb forces, are not yet fully understood. Using ensemble (19) or (32) as their model, the physical properties of these systems can now be probed further and earlier experimental observations may be explained. For example, it has been experimentally observed that the peak-spacing statistics for an irregular quantum dot in Coulomb blockade regime undergoes a crossover from Wigner-Dyson to Gauss-

ian behavior as the strength of electron-electron interaction increases [1,2]. Within our formulation, the observed behavior can be explained as follows. The single electron dynamics inside a quantum dot of irregular shape is chaotic; the level statistics of single particle Hamiltonian V can therefore be modeled by the Wigner-Dyson behavior [3]. The addition of more electrons switches the potential U, however, due to nonuniform electronic density during initial stages, the correlation between various matrix elements of U need not be the same. The statistical behavior of the quantum dot can therefore be described by an ensemble given by Eq. (19). As electron density increases, U dominates over V and the level-statistics approaches the behavior of antisymmetric matrices. It is already known that the nearest-neighbor spacing distribution for noncentral spacings in the spectrum of antisymmetric Hermitian matrices behaves like a Gaussian distribution [5]. The observed Gaussian behavior of the peak spacings in the strong interaction limit is therefore well in agreement with theoretical expectations. Using ensemble (19) as a model for the quantum dot Hamiltonian, the behavior of peak spacings in the intermediate regime can be predicted to be similar to that of a BE appearing during a crossover from GOE→antisymmetric ensembles. A detailed quantitative analysis of such cases is still in progress and will be published elsewhere.

In general, the size dependence of the parameter $Y-Y_0$ and the local mean level spacing Δ_l for a system with e-e interaction is different from the noninteracting systems. In interacting systems, therefore, the critical point of level statistics, given by condition $\Lambda=N-$independent, can occur at a disorder strength (or energy) different from the one for noninteracting systems. This is consistent with the results given by renormalization group techniques [15] which show that the introduction of interactions into quantum dots can produce phase transitions in the limit of weak disorder, leading to behavior qualitatively different from the noninteracting case. Further, for one-dimensional noninteracting systems, it is known that the wave functions are localized even in a weak disorder limit. However, our formulation indicates that the fermionic interaction may lead to extended states even in the one-dimensional disordered system; the implication is in agreement with earlier studies in this context [16].

V. CONCLUSION

In summary we show that the level statistics of disordered systems with interactions is governed by a single parameter, namely, the rescaled complexity parameter Λ. Note the level statistics of noninteracting systems can also be described in terms of Λ [10]. However the introduction of interactions modifies the dependence of Λ on system parameters which can significantly affect the location of the critical point for the phase transitions and corresponding level statistics. Our study also reveals a deep level of universality underlying physical systems, namely, the Brownian ensembles as the statistical backbone of both interacting as well as noninteracting systems. This universality should be explored in full detail as it may reveal many new connections among a wide range of complex systems and can be helpful in theoretical formulation of many of their physical properties.

APPENDIX A: THE CHANGE OF EIGENVALUES AND EIGENFUNCTIONS

The eigenvalue equation of a complex Hermitian matrix H is given by $HU=U\Lambda$ with Λ as the matrix of eigenvalues λ_n and U as the eigenvector matrix, unitary in nature. As is obvious, a slight variation of the matrix elements of H will, in general, lead to variation of both the eigenvalues as well as the eigenvectors and associated rates of change can be obtained as follows (see Appendices A–E of Ref. [10] for more details).

As $\lambda_n=\Sigma_{i,j}U_{ni}H_{ij}U_{nj}^*$, the rate of change of λ_n with respect to $H_{kl;s}$ (with s referring to real, $s=1$, and imaginary, $s=2$, parts of H_{kl}) can be given as

$$\frac{\partial\lambda_n}{\partial H_{kl;s}}=\frac{i^{s-1}}{g_{kl}}[U_{ln}U_{kn}^*-(-1)^sU_{ln}^*U_{kn}], \tag{A1}$$

where $g_{kl}=1+\delta_{kl}$. This can further be used to obtain the following relations:

$$\sum_{k\leqslant l}\sum_{s=1}^{2}\frac{\partial\lambda_n}{\partial H_{kl;s}}H_{kl;s}=\sum_{k,l}H_{kl}U_{ln}U_{kn}^*=\lambda_n \tag{A2}$$

and

$$\sum_{k\leqslant l}g_{kl}\sum_{s=1}^{2}\frac{\partial\lambda_n}{\partial H_{kl;s}}\frac{\partial\lambda_m}{\partial H_{kl;s}}=2\delta_{mn}. \tag{A3}$$

As obvious from Eq. (A1), the second order change of an eigenvalue with respect to a matrix element requires a knowledge of the rate of change of one of the eigenvector components with respect to H_{kl}. The latter can again be obtained by using the eigenvalue equation,

$$\frac{\partial U_{pn}}{\partial H_{kl;s}}=\frac{i^{s-1}}{g_{kl}}\sum_{m\neq n}\frac{1}{\lambda_n-\lambda_m}U_{pm}[U_{km}^*U_{ln}+(-1)^{s+1}U_{lm}^*U_{kn}]. \tag{A4}$$

Now by differentiating Eq. (A1) with respect to $H_{kl;s}$ and by using Eq. (A4) we can show that

$$\sum_{k\leqslant l}g_{kl}\sum_{s=1}^{2}\frac{\partial^2\lambda_n}{\partial H_{kl;s}^2}=2\beta\sum_{m}\frac{1}{\lambda_n-\lambda_m}. \tag{A5}$$

For the real-symmetric case, the corresponding relations can be obtained by using $U^+=U^T$ (as eigenvector matrix is now orthogonal) in Eq. (A1) and taking $H_{ij;2}=0$ for all values of i,j.

APPENDIX B: SOLUTION OF EQ. (38)

According to theory of partial differential equations (PDE) [14], the general solution of a linear PDE

$$\sum_{i=1}^{M} P_i(x_1,x_2,\ldots,x_M)\frac{\partial Z}{\partial x_i} = R \tag{B1}$$

is $F(u_1,u_2,\ldots,u_n)=0$ where F is an arbitrary function and $u_i(x_1,x_2,\ldots,x_n,Z)=c_i$ (a constant), $i=1,2,\ldots,n$ are independent solutions of the following equation:

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \cdots \frac{dx_k}{P_k} = \cdots \frac{dx_M}{P_M} = \frac{dZ}{R}. \tag{B2}$$

Note the function F being arbitrary, it can also be chosen as

$$F \equiv \sum_j (u_j - c_j) = 0. \tag{B3}$$

The equations for various y_j in the set of Eq. (38) are of the same form as Eq. (B2) and, therefore, can be solved as described above. Let us first consider the equation for y_1; its general solution can be given by a relation $F(u_1,u_2,\ldots,u_M)=0$ where function F is arbitrary and u_j are the functions of M parameters of set b such that $u_j(\{b\},y_1)=c_j$ (with c_j's as constants). The functions u_j are the independent solutions of the equation

$$\frac{db_{p(1)}}{f_{p(1)}} = \cdots = \frac{db_{p(2)}}{f_{p(2)}} = \cdots \frac{db_{p(r)}}{f_{p(r)}} = \cdots = dy_1, \tag{B4}$$

where the equality between ratios is implied for all possible combinations $p(r)$ of r terms, $r=1\rightarrow n$, with M as the total number of combinations. It is easy to see that each of the above ratios is equal to $\Sigma_{r,p(r)} z_p^{(1)} db_p / \Sigma_{r,p(r)} z_p^{(1)} f_p$ where $z_p^{(1)}$ are arbitrary functions. Equation (B4) can therefore be rewritten as

$$dy_1 = \frac{\sum_{r,p(r)} z_p^{(1)} db_p}{\sum_{r,p} z_p^{(1)} h_p}. \tag{B5}$$

A solution, say u_1 of Eq. (B5), or alternatively Eq. (B4), can now be obtained by choosing the functions $z_p^{(1)}$ such that the right side of the above equation becomes an exact differential:

$$u_1 \equiv y_1 - \sum_{r,p(r)} \int db_p z_p^{(1)} G_1 = \text{const}, \tag{B6}$$

where $G_1=[\Sigma_{r,p(r)} z_p^{(1)} h_p]^{-1}$. The general solution for y_1 can therefore be given by a combination of all possible functions u obtained by using an arbitrary set of z functions. It can be shown that each such solution differs from the other only by a constant: $u_j=u_i+\text{const}$ (this is due to equality of the two ratios obtained by choosing two different sets $z^{(1)}$ of the functions). The y_1 can therefore be written as follows:

$$y_1 = \sum_{r,p(r)} \int db_p z_p^{(1)} G_1 + \text{const} \tag{B7}$$

which gives Eq. (40) for $k=1$.

The set of Eqs. (38) can similarly be solved for other y_j ($j>2$). For example, the solution of Eq. (38) for y_k can be given by the function $F_k(v_1,\ldots,v_M)=0$ where $v_j(\{b\},y_k)=\text{const}$ are the independent solutions of following equality:

$$\frac{db_{p(1)}}{f_{p(1)}} = \cdots = \frac{db_{p(2)}}{f_{p(2)}} = \cdots \frac{db_{p(r)}}{f_{p(r)}} = \cdots \frac{dy_k}{0}. \tag{B8}$$

A solution, say v_1, of Eq. (B8) can now be given as

$$v_1 \equiv y_k - \sum_{r,p(r)} \int db_p z_p^{(k)} = \text{const}, \tag{B9}$$

where $z_p^{(k)}$ are arbitrarily chosen M functions which satisfy the condition

$$\sum_{r,p(r)} z_p^{(k)} f_p = 0. \tag{B10}$$

As is obvious, one possible choice for $z^{(k)}$ functions satisfying the above condition is $z_p^{(k)}=0$ for all $p(r)$ which gives $y_k=\text{const}$.

As each solution of Eq. (B8) is different from the other only by a constant, the general solution for y_k, $k>1$, can now be given as

$$y_k = \sum_{r,p(r)} \int db_p z_p^{(k)} + \text{const}. \tag{B11}$$

Equation (B7) and Eq. (B11) together give the set of Eq. (40).

APPENDIX C: EXAMPLE FOR QUANTUM HALL CASE

Within independent Landau level approximation, the quantum Hall ensemble can be described by the probability density $\rho(H)$ given by Eq. (1) with coefficients b given by Eqs. (57) and (58). However, it can further be simplified by choosing the origin of energy at ϵ_0 which makes $b_\mu=\langle H_\mu\rangle=0$. The matrix element distribution in the quantum Hall case can now be described by a probability density

$$\tilde{\rho}(H,b) = C \exp\left[- \sum_{\mu_1,\mu_2=1}^{M} b_{\mu_1\mu_2} H_{\mu_1} H_{\mu_2} \right] \tag{C1}$$

with C as the normalization constant

$$C^{-1} = \int dH e^{-\Sigma_{\mu_1\mu_2} b_{\mu_1\mu_2} H_{\mu_1} H_{\mu_2}} = \left(\prod_{s=1}^{2} \text{Det } B_s \right)^{-1/2}. \tag{C2}$$

Here $B^{(s)}$ is the matrix of coefficients $b_{\mu_1\mu_2}$. The parameters $b_{\mu_1\mu_2}$ are related to second order correlation $\langle H_{\mu_1} H_{\mu_2}\rangle$:

$$\langle H_{\mu_1} H_{\mu_2}\rangle = \int H_{\mu_1} H_{\mu_2} \tilde{\rho} dH \tag{C3}$$

$$= \frac{\eta_{\mu_1\mu_2}}{2} \frac{\partial \ln C}{\partial b_{\mu_1\mu_2}}, \tag{C4}$$

where $\eta_{\mu_1\mu_2}=2$ and 1 for pairs $\{\mu_1\}=\{\mu_2\}$ and $\{\mu_1\}\neq\{\mu_2\}$, respectively. Let $Q^{(s)}$ be the matrix with its elements as the correlations between elements of H, that is, $(Q^{(s)})_{\mu_1\mu_2}=(4/\eta_{\mu_1\mu_2})\langle H_{\mu_1}H_{\mu_2}\rangle$. By using Eq. (C2) in Eq. (C4), we get

$$Q^{(s)}_{\mu_1\mu_2}=\frac{\partial \ln \mathrm{Det}[B^{(s)}]}{\partial b_{\mu_1\mu_2}}=\frac{\mathrm{Cof}(B_{\mu_1\mu_2})}{\mathrm{Det}[B^{(s)}]}=(B^{(s)})^{-1}_{\mu_2\mu_1}, \quad \text{(C5)}$$

where $\mathrm{Cof}(B^{(s)}_{\mu_1\mu_2})$ implies the cofactor of the element $B_{\mu_1\mu_2}$ in the matrix B_s. This implies $Q^{(s)}=(B^{(s)})^{-1}$. The matrix $B^{(s)}$ for the QH case can therefore be obtained by inverting the correlation matrix $Q^{(s)}=\{Q^{(s)}_{\mu_1\mu_2}\}$.

Let us consider the case $N=2$. Using the H_μ notation, various components of matrix elements can now be denoted as $H_1=H_{11;1}$, $H_2=H_{12;1}$, $H_3=H_{22;1}$, $H_4=H_{12,2}$. Following Eq. (60), the correlation matrix $Q^{(1)}$ in this case is a 3×3 matrix

$$Q^{(1)}=\begin{pmatrix} a & o & 2ax \\ o & a/2x_1 & 0 \\ 2ax & 0 & a \end{pmatrix}, \quad \text{(C6)}$$

where $a=(V_0^2/2l_cL_y\alpha\sqrt{2\pi})$, $x=e^{-1/2\alpha^2}$ and $x_1=e^{\alpha^2/2}$. By using the relation (C5), $B^{(1)}$ can be given as

$$B^{(1)}=\frac{1}{a(1-4x^2)}\begin{pmatrix} 1 & o & -2x \\ o & 8x_1(1-4x^2) & 0 \\ -2x & 0 & 1 \end{pmatrix}. \quad \text{(C7)}$$

Due to the Hermitian nature of H, only its off-diagonal elements have imaginary parts. For $N=2$ case, therefore, $B^{(2)}$ is just a 1×1 matrix, corresponding to correlation $\langle H_4H_4\rangle$ with $\mu_4\equiv(12;2)$: $B^{(2)}=b_{44}=(2\langle H_4H_4\rangle)^{-1}$. The parameter set y for this case can now be obtained by solving the condition $A_k=\Sigma_{i,j=1}^4 f_{ij}(\partial y_k/\partial b_{ij})=\delta_{k1}$, where $f_{ij}=\gamma b_{ij}-(1/2)\Sigma_{k=1}^4 g_k c_{ik}c_{jk}$; here $g_k=2$ for odd k and $g_k=1$ for even k. As discussed in Appendix B, a solution of the condition can be given as

$$\frac{db_{11}}{f_{11}}=\frac{db_{12}}{f_{12}}=\frac{db_{13}}{f_{13}}=\frac{db_{22}}{f_{22}}=\frac{db_{31}}{f_{31}}=\frac{db_{23}}{f_{32}}=\frac{db_{33}}{f_{33}}=\frac{db_{44}}{f_{44}}=\frac{dy_k}{\delta_{k1}}. \quad \text{(C8)}$$

The above equations can now be solved by making the ratio exact differential [following from Eq. (B5) of Appendix B],

$$\frac{dF_k}{G_k}=\frac{dy_k}{\delta_{k1}}, \quad \text{(C9)}$$

where $dF_k\equiv\Sigma_{i,j}z_{ij}^{(k)}db_{ij}$ and $G_k\equiv\Sigma_{i,j}z_{ij}^{(k)}f_{ij}$ with $z^{(k)}$ as arbitrary functions. Using Eq. (C7), the G_k can be shown to be

$$\begin{aligned} G_k=&\frac{4}{(1-4x^2)^2a^2}[(z_{11}^{(k)}+z_{33}^{(k)})[(\gamma a-4)-4x^2(\gamma a+1)] \\ &+(z_{13}^{(k)}+z_{31}^{(k)})4x(8-\gamma a(1-4x^2)) \\ &+16z_{22}^{(k)}a^{-2}x_1(\gamma a-16x_1)-z_{44}^{(k)}ax_1^{-1}(2\gamma x_1+a)/2]. \end{aligned} \quad \text{(C10)}$$

Similarly

$$\begin{aligned} dF_k\equiv\sum_{ij}z_{ij}^{(k)}db_{ij}=&[(z_{11}^{(k)}+z_{33}^{(k)})dx_2-2(z_{13}^{(k)}+z_{31}^{(k)})d(xx_2) \\ &+2(4z_{22}^{(k)}-z_{44}^{(k)})d(x_1/a)], \end{aligned} \quad \text{(C11)}$$

where $x_2=a^{-1}(1-4x^2)^{-1}$. As both dF_1 and G_1 are functions of l_c and σ (through a and α), y_1 will turn out to be a function of parameter l_c and σ,

$$y_1=\sum_{ij}\int db_{ij}\frac{z_{ij}^{(1)}}{G_1}=\int\frac{dF_1(l_c,\sigma)}{G_1(l_c,\sigma)}. \quad \text{(C12)}$$

Proceeding similarly for $y_kk>2$, a solution for y_k can be given as

$$y_k=\sum_{ij}\int db_{ij}z_{ij}^{(k)}, \quad \text{(C13)}$$

where $z^{(k)}$ satisfy the conditions $G_k=0$.

Note the condition $G_2=0$ is satisfied for a following choice of $z^{(2)}$: $z_{11}^{(2)}=-z_{33}^{(2)}$, $z_{13}^{(2)}=-z_{31}^{(2)}$, and $z_{22}^{(2)}=z_{44}^{(2)}=0$. The y_2 for this choice turns out to be a constant. Similarly the condition $G_3=0$ can be satisfied for a following choice of $z^{(3)}$: $(z_{11}^{(3)}+z_{33}^{(3)})/(z_{13}^{(3)}-z_{31}^{(3)})=[4x8-\gamma a(1-4x^2)]/[(\gamma a-4)-4(\gamma a+1)x^2]$ and $z_{22}^{(3)}/z_{44}=a^3x_1^3(\gamma+ax_1/32)/64(\gamma ax_1-16)$. Using these z values in Eq. (C13), one can obtain y_3 as a function of σ and l_c. Note although y_3 varies with changing σ and l_c, however $\Sigma_{ij}f_{ij}(\partial y_3/\partial b_{ij})=0$ and therefore y_3 does not affect the evolution of $\rho(H)$.

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